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A Note on Girard Bimodules†

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Any (involutive) quantale is embeddable into the quantale of $\bigvee$-endomorphisms of a Girard bimodule over Q. Any Q-module (Q-valued module) is representable as a concrete submodule of the simple involutive quantale of $\bigvee$-endomorphisms of a Girard bimodule over Q.

Around 1985, the notion of a quantale arose in pure algebra, i.e., a complete lattice Q with an associative binary operation $\cdot$ that distributes over arbitrary joins from left and right. The study of such algebras goes back to the work of Ward and Dilworth on residuated lattices. It has become a useful tool in studying noncommutative topology, linear logic, and C*-algebra theory. Motivated by the notion of a Girard quantale, a Girard bimodule M is defined exactly as a complete lattice with a duality such that it is both a right and left Q-module and a weak compatibility condition between the module actions is satisfied. In this paper we present the thesis that Girard bimodules form a natural context for a concrete representation of (involutive) quantales.

Our main motivating source was ref. 8, where the investigation of I-simple involutive quantales is developed. Such involutive quantales could play a similar role as points in topological spaces or irreducible representations in C*-algebras [4] or Hilbert modules [3]. This note is closely related to refs. 5–7, where the interested reader can find unexplained terms and notation concerning the subject. For facts concerning quantales and Q-modules in general we refer to refs. 10 and 11. For motivating examples concerning Q-modules we recommend refs. 1 and 9.

† This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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The paper is organized as follows. Section 1 introduces the notion of a left (right) Q-module and a Girard bimodule. The motivation for studying such a structure is given. We prove that any (involutive) quantale is embeddable into the simple involutive quantale of $\bigvee$-endomorphisms of a Girard bimodule over Q.

In Section 2 we show that any Q-module is representable as a concrete submodule of the simple involutive quantale of $\bigvee$-endomorphisms of a Girard bimodule over Q, i.e., any Q-module can be equipped with a structure of a Q'-valued module such that Q is a subquantale of Q'. Moreover, any Q-valued module is representable as a concrete submodule of the simple involutive quantale of $\bigvee$-endomorphisms of a Girard bimodule over Q such that the representation is inner product-preserving.

We begin by establishing the symbols and notation in this paper.

A *quantale* is a complete lattice Q with an associative binary multiplication satisfying

$$x \cdot \bigvee_{i \in I} x_i = \bigvee_{i \in I} x \cdot x_i \qquad \text{and} \qquad (\bigvee_{i \in I} x_i) \cdot x = \bigvee_{i \in I} x_i \cdot x$$

for all $x, x_i \in Q$, $i \in I$ (I is a set). Since $a \cdot$ —: $Q \to Q$ and —$\cdot\, a$: $Q \to Q$ preserve arbitrary suprema, they have right adjoint we shall denote by $a \to_r$ —: $Q \to Q$ and $a \to_l$ —: $Q \to Q$, respectively. 1 denotes the greatest element of Q, 0 is the smallest element of Q. A quantale Q is said to be *unital* if there is an element $e \in Q$ such that $e \cdot a = a = a \cdot e$ for all $a \in Q$. A *subquantale* Q' of a quantale Q is a subset of Q closed under $\bigvee$ and $\cdot$.

By a *morphism of quantales* will be meant a $\bigvee$- and $\cdot$-preserving mapping $f: Q \to Q'$.

A nontrivial quantale Q is said to be *simple* if any surjective homomorphism $Q \to Q'$ is either an isomorphism or a constant morphism.

By the *quantale* $\mathfrak{Q}(M)$ *of endomorphisms of the sup-lattice* M will be meant the simple unital quantale of sup-preserving mappings from M to itself, with the supremum given by the pointwise ordering of mappings, with the multiplication corresponding to the composition of mappings, and with the unit given by the identity mapping.

1. *Q*-MODULES

Definition 1.1[1, 11]. Let Q be a quantale. A *left module over* Q (briefly, a left Q-module) is a sup-lattice M, together with a *module action* -●-: $Q \times M \to M$ satisfying

$$(a \cdot b) \bullet m = a \bullet (b \bullet m) \tag{M1}$$

$$(\bigvee S) = \bigvee\{s \bullet m: s \in S\} \tag{M2}$$

$$a \bullet \bigvee X = \bigvee\{a \bullet x: x \in X\} \tag{M3}$$

for all $a, b \in Q$, $m \in M$, $S \subseteq Q$, $X \subseteq M$. So we have two maps $- \twoheadrightarrow_l -: M \times M \to Q$ and $- \twoheadrightarrow_r -: Q \times M \to M$ such that, for all $a \in Q$, m, n, $\in M$, $a \bullet m \leq n$ iff $a \leq m \twoheadrightarrow_l n$ iff $m \leq a \twoheadrightarrow_r n$. M is called a *unital* Q-module if Q is a unital quantale with the unit e and $e \bullet m = m$ for all $m \in M$.

Let M and N be modules over Q and let $f: M \to N$ be a sup-lattice homomorphism, f is a *module homomorphism* if $f(a \bullet m) = a \bullet f(m)$ for all $a \in Q$, $m \in M$.

Note first that if M is a sup-lattice, then M is a left $Q(M)$-module such that $f \bullet m = f(m)$ for all $f \in Q(M)$ and all $m \in M$. Second, we may dually define the notion of a (unital) *right Q-module* with a right multiplication $\diamondsuit$. We then have two maps $- \twoheadrightarrow_R -: M \times M \to Q$ and $- \twoheadrightarrow_L -: Q \times M \to M$ such that, for all $a \in Q$, $m, n \in M$, $m \diamondsuit a \leq n$ iff $a \leq m \twoheadrightarrow_R n$ iff $m \leq a \twoheadrightarrow_L n$. Moreover, all propositions stated for left Q-modules are valid in a dualized form for right Q-modules and conversely.

A *bimodule* M over a quantale Q is both a left Q-module with action $\bullet$ and a right Q-module with action $\diamondsuit$.

For every left Q-module M we have a *dual* right Q-module M^{op} with multiplication given by $a \twoheadrightarrow_r m$. Similarly, for every right Q-module N we have a *dual* left Q-module M^{op}. Note that, if M is unital, then M^{op} is unital.

Note that, for every involutive quantale Q and any left Q-module M we have a right Q-module M^* with the same lattice structure and a right multiplication $\bullet_*$ defined by $m \bullet_* a = a^* \bullet m$. Following ref. 2, we shall say that a left Q-module is *involutive* if $a \bullet m = (a^* \twoheadrightarrow_r m^\perp)^\perp$. This is equivalent to the fact that the quantale morphism $f: Q \to \mathfrak{Q}(M)$ defined by $f(a)(m) = a \bullet m$ is a morphism of involutive quantales.

We shall denote by Q-**Mod**, resp. **Mod**-Q, the category of left Q-modules resp., right Q-modules.

Let us make the following elementary observation. Evidently, for a quantale Q, any left Q-module M is a left unital $Q[e]$-module with $Q[e]$ defined as in ref. 6 and the multiplication $\bullet_e$ defined by

$$(a \vee \varepsilon) \bullet_e m = \begin{cases} a \bullet m & \text{if} \quad \varepsilon = 0 \\ a \bullet m \vee m & \text{if} \quad \varepsilon = e \end{cases}$$

So we may always assume that any left quantale module is unital over a unital quantale.

Note that, for any quantale Q, any left Q-module M and all $n \in M$, the antitone maps $- \twoheadrightarrow_l n: M \to Q$ and $- \twoheadrightarrow_r n: Q \to M$ form a Galois connection between Q and M since, for all $a \in Q$ and all $m \in M$, $a \leq (a \twoheadrightarrow_r n) \twoheadrightarrow_l n$ and $m \leq (m \twoheadrightarrow_l n) \twoheadrightarrow_r n$. So we have that $m \twoheadrightarrow_l n = ((m \twoheadrightarrow_l n)$

$\twoheadrightarrow_r n) \twoheadrightarrow_l n$ and $a \twoheadrightarrow_r n = ((a \twoheadrightarrow_r n) \twoheadrightarrow_l n) \twoheadrightarrow_r n$. The same holds for the antitone maps $- \twoheadrightarrow_R n\colon M \to Q$ and $- \twoheadrightarrow_L n\colon Q \to M$.

Definition 1.2. Let Q be a quantale. A *Girard bimodule* over Q is a bimodule over Q with a duality ${}^\perp\colon M \to M$ such that

$$a \bullet m = (a \twoheadrightarrow_L m^\perp)^\perp$$

for all $a \in Q$, $m \in M$.

Note that, in a Girard bimodule M, we have $m \diamond a = (a \twoheadrightarrow_r m^\perp)^\perp$. Namely, $m \diamond a \leq n$ iff $m \leq a \twoheadrightarrow_L n$ iff $m \leq (a \bullet n^\perp)^\perp$ iff $m^\perp \geq a \bullet n^\perp$ iff $n^\perp \leq a \twoheadrightarrow_r m^\perp$ iff $(a \twoheadrightarrow_r m^\perp)^\perp \leq n$. Moreover, if Q' is a a subquantale of Q, we have that M is a Girard bimodule over Q'.

Lemma 1.3. Let Q be a quantale, M a left Q-module with a duality ${}^\perp$. Then M is a Girard bimodule with the right action $\diamond\colon M \times Q \to M$ defined by $m \diamond a = (a \twoheadrightarrow_r m^\perp)^\perp$.

Proof. Let $a, b \in Q$, $m \in M$. We have $(m \diamond b) \diamond a \leq n$ iff $(a \twoheadrightarrow_r (b \twoheadrightarrow_r m^\perp))^\perp \leq n$ iff $b \twoheadrightarrow_r m^\perp \geq a \bullet n^\perp$ iff $m^\perp \geq b \bullet (a \bullet n^\perp)$ iff $m^\perp \geq (b \cdot a) \bullet n^\perp$ iff $(b \cdot a) \twoheadrightarrow_r m^\perp \geq n^\perp$ iff $((b \cdot a) \twoheadrightarrow_r m^\perp)^\perp \leq n$ iff $m \diamond (b \cdot a) \leq n$. Similarly, we can prove that $m \diamond -\colon Q \to M$ and $- \diamond a\colon Q \to M$ are $\bigvee$-preserving maps for all $a \in Q$ and $m \in M$, i.e., we have a right module action $\diamond$. Let us prove that $a \bullet m = (a \twoheadrightarrow_L m^\perp)^\perp$. We have $n \geq (a \twoheadrightarrow_L m^\perp)^\perp$ iff $n^\perp \leq a \twoheadrightarrow_L m^\perp$ iff $n^\perp \diamond a \leq m^\perp$ iff $(a \twoheadrightarrow_r n)^\perp \leq m^\perp$ iff $a \twoheadrightarrow_r n \geq m$ iff $a \bullet n \leq n$. ■

Definition 1.4 [10]. Let Q be a quantale. An element $d \in Q$ is called a *dualizing element* iff for all $a \in Q$, we have $(a \to_r d) \to_l d = a = (a \to_l d) \to_r d$. An element $c \in Q$ is said to be *cyclic* iff $a \to_l c = a \to_r c$ for all $a \in Q$. A quantale is called a *Girard quantale* iff it has a cyclic dualizing element d.

Note that Girard quantales can be thought of as "Boolean quantales" [10]. Then, by the next theorems, Girard bimodules can be thought of as "Boolean modules" since we shall show that any Girard quantale is a Girard bimodule and that any quantale (involutive) quantale is an (involutive) subquantale of $\mathfrak{Q}(M)$ for a suitable Girard bimodule.

Proposition 1.5. Let Q be a unital quantale with a duality ${}^\perp$ such that $a \cdot m = (a \to_l m^\perp)^\perp$ for all $a, m \in Q$, i.e., Q is a Girard bimodule over Q. Then Q is a Girard quantale. Moreover, any Girard quantale is a Girard bimodule over itself.

Proof. Let us put $d = e^\perp$. We shall show that d is both cyclic and dualizing. We have $a \to_r e^\perp = (e \cdot a)^\perp = (a \cdot e)^\perp = a \to_l e^\perp$. Similarly,

$(a \to_r e^\perp) \to_l e^\perp = [(a \to_r e^\perp) \cdot e]^\perp = [(e \cdot a)^\perp \cdot e]^\perp = a^{\perp\perp} = a$ and $(a \to_l e^\perp) \to_r e^\perp = a$. The rest follows from Proposition 6.1.2 of ref. 10. ■

Theorem 1.6. Let Q be quantale. Then we have a quantale embedding i_Q: $Q \to \mathfrak{Q}(M)$ such that M is a Girard bimodule over Q.

Proof. By ref. 6, we know that any quantale Q is always embeddable into an unital quantale. So we may assume that Q is a unital quantale with the unit e. Let us define $\bullet$: $Q \times (Q \times Q^{op}) \to Q \times G^{op}$ and $\diamond$: $(Q \times Q^{op}) \times Q \to Q^{op}$ as follows: $a \bullet (m, n) = (a \cdot m, a \to_l n)$ and $(m, n) \diamond a = (m \cdot a, a \to_r n)$ for all $a, m, n \in Q$. Then, when we define a duality $^\perp$: $Q \times Q^{op} \to Q \times Q^{op}$ by $(m, n)^\perp = (n, m)$, we have $(a \twoheadrightarrow_L (m, n)^\perp)^\perp = (a \twoheadrightarrow_L (n, m))^\perp = (\bigvee\{p: p \cdot a \le n\}, \bigwedge\{q: m \le a \to_r q\})^\perp = (\bigwedge\{q: m \le a \to_r q\}, \bigvee\{p: p \cdot a \le n\}) = (a \cdot m, a \to_l n) = a \cdot (m, n)$. Let us put $M = Q \times Q^{op}$. Then M is a Girard bimodule and we have a morphism i_Q: $Q \to \mathfrak{Q}(M)$ of quantales defined by $i_Q(a)(m, n) = a \bullet (m, n)$. Let $i_Q(a) = i_Q(b)$. Then $(a, a \to_l e) = i_Q(a)(e, e) = i_Q(b)(e, e) = (b, b \to_l e)$, i.e., $a = b$ and i_Q is a quantale embedding. ■

Let φ: $M \to N$ be a join-preserving morphism of $\bigvee$-lattices. Then we have a right adjoint map $\varphi^{\dashv}$: $N \to M$ preserving arbitrary meets. If M is a $\bigvee$-lattice with a duality $^\perp$, there is an involution on $\mathfrak{Q}(M)$ given by $\varphi^*(m) = (\varphi^{\vdash}(m^\perp))^\perp$.

Theorem 1.7 [6]. Let Q be an involutive quantale. Then we have an involutive quantale embedding I_Q: $K \to \mathfrak{Q}(M)$ such that M is an involutive Girard bimodule over Q.

Proof. Again, by ref. 6, we know that any involutive quantale Q is always involutively embeddable into a unital involutive quantale. So we may assume that Q is an involutive unital quantale with the unit e. Let us define $\bullet$: $Q \times (Q \times Q^{op}) \to Q \times Q^{op}$ and $\diamond$: $(Q \times Q^{op}) \times Q \to Q \times Q^{op}$ as follows: $a \bullet (m, n) = (a \cdot m, a^* \to_r n)$ and $(m, n) \diamond a = (a^* \cdot m, a \to_r n)$ for all $a, m, n \in Q$. Then we have $(a \twoheadrightarrow_L (m, n)^\perp)^\perp = (a \twoheadrightarrow_L (n, m))^\perp = (\bigvee\{p: p \cdot a^* \le n\}, \bigwedge\{q: m \le a \to_r q\})^\perp = (\bigwedge\{q: a \cdot m \le q\}, \bigvee\{p: p \le a^* \to_r n\}) = (a \cdot m, a^* \to_i n) = \bullet (m, n)$. Let us put $M = Q \times Q^{op}$. Then M is a Girard bimodule. Moreover, we have $(a^* \twoheadrightarrow_r (m, n)^\perp)^\perp = (a^* \twoheadrightarrow_r (n, m))^\perp = (\bigvee\{p: a^* \cdot p \le n\}, \bigwedge\{q: m \le (a^*)^* \to_l q\})^\perp = (\bigwedge\{q: a \cdot m \le q\}, \bigvee\{p: p \le a^* \to_r n\}) = (a \cdot m, a^* \to_r n) = a \bullet (m, n)$. We then have a morphism I_Q: $Q \to \mathfrak{Q}(M)$ of involutive quantales defined by $I_Q(a)(m, n) = a \bullet (m, n)$ that is evidently a quantale embedding. ■

2. *Q*-VALUED MODULES

In this section we shall show that any Q-valued module M is representable together with the quantale Q as a concrete submodule and a concrete subquantale of the simple involutive quantale $\mathfrak{Q}(S)$ for a suitable Girard bimodule S such that the module action coincides with the usual composition of mappings in $\mathfrak{Q}(S)$.

Definition 2.1 Let Q be a unital quantale, M a right (left) Q-module. We say that M is *right* (*left*) *Q-valued* if M is equipped with a map $\langle —, — \rangle$: $M \times M \to Q$, called the *inner product*, such that for all $a \in Q$, m, $n \in M$, and $m_i \in M$, where $i \in I$, the following conditions are satisfied:

(i) $\langle m, n \rangle \diamondsuit a = \langle m, n \diamondsuit a \rangle$ $(a \cdot \langle m, n \rangle = \langle a \bullet m, n \rangle)$.
(ii) $\bigvee_{i \in I} \langle m_i, n \rangle = \langle \bigvee_{i \in I} m_i, n \rangle$.
(iii) $\bigvee_{i \in I} \langle m, m_i \rangle = \langle m, \bigvee_{i \in I} m_i \rangle$.
(iv) $\langle —, m \rangle = \langle —, n \rangle$ $(\langle m, — \rangle = \langle n, — \rangle)$ implies $m = n$.

A basic example of a right (left) Q-valued module is $M = Q$ with the inner product $\langle m, n \rangle = m \cdot n$; another example is any right (left) complete ideal of Q with the same Q-valued inner product.

Definition 2.2. A *representation of a right (left) Q-module M* is a pair of maps (Φ, φ) such that S is a $\vee$-lattice with a duality, φ: $Q \to \mathfrak{Q}(S)$ is a representation of the quantale Q, and Φ: $M \to \mathfrak{Q}(S)$ is a $\bigvee$-preserving one-to-one map such that

$$\Phi(m \diamondsuit a) = \Phi(m) \circ \varphi(a) \qquad [\Phi(a \bullet m) = \varphi(a) \circ \Phi(m)]$$

for all $a \in Q$ and $m \in M$. A representation of a right (left) Q-valued module M is called *inner product-preserving* if $\varphi(\langle m, n \rangle) = \Phi^*(m) \circ \Phi(n)$ for all m, $n \in M$; here Φ^*: $M \to \mathfrak{Q}(S)$ is a $\bigvee$-preserving one-to-one map such that $\Phi^*(m) = \Phi(m)^*$.

Theorem 2.3. Let Q be a unital quantale, M a right Q-module. Then we have a representation (Φ, φ) such that S is a Girard bimodule over Q.

Proof. First we shall show that any right Q-module M can be always embedded into a certain quantale $\Lambda(M)$. Denote by $N = Q \times M$ the right Q-module

We can identify each element $m \in M$ with the operator $m^{\sim}$: $Q \to M$ defined by $a \mapsto m \diamondsuit a$ and each element $b \in Q$ with the operator $b^{\smile}$: $Q \to Q$ given by $b^{\smile}(c) = b \cdot c$. Note that $b^{\smile} \circ c^{\smile} = (b \cdot c)^{\smile}$. Let $\Lambda(M)$ be the subset of $\mathfrak{Q}(N)$ consisting of all $\bigvee$-preserving maps on N that can be represented by matrices of the form

$$\begin{bmatrix} f & g \\ h & \psi \end{bmatrix}$$

$f \in \mathfrak{Q}(\mathrm{Q})$, $g \in \mathbf{Sup}(M, Q)$, $h \in \mathbf{Sup}(Q, M)$, $\psi \in \mathfrak{Q}(M)$. Note that the composition of such matrices is defined as follows:

$$\begin{bmatrix} f_1 & g_1 \\ h_1 & \psi_1 \end{bmatrix} \circ \begin{bmatrix} f_2 & g_2 \\ h_2 & \psi_2 \end{bmatrix} = \begin{bmatrix} f_1 \circ f_2 \vee g_1 \circ h_2 & f_1 \circ g_2 \vee g_1 \circ \psi_2 \\ h_1 \circ f_2 \vee \psi_1 \circ h_2 & h_1 \circ g_2 \vee \psi_1 \circ \psi_2 \end{bmatrix}$$

and evidently coincides with the usual composition of maps; here the action is defined as

$$\begin{bmatrix} f & g \\ h & \psi \end{bmatrix}(a, m) = (f(a) \vee g(m), h(a) \vee \psi(m))$$

Similarly, the join of such matrices is defined by

$$\bigvee_{i \in I} \begin{bmatrix} f_i & g_i \\ h_i & \psi_i \end{bmatrix} = \begin{bmatrix} \bigvee_{i \in I} f_i & \bigvee_{i \in I} g_i \\ \bigvee_{i \in I} h_i & \bigvee_{i \in I} \psi_i \end{bmatrix}$$

Then $\Lambda(M)$ is a unital subquantale of $\mathfrak{Q}(N)$; the unit id_N can be represented by the matrix

$$\begin{bmatrix} \mathrm{id}_Q & 0 \\ 0 & \mathrm{id}_M \end{bmatrix}$$

$\Lambda(M)$ will enable us to exploit the representation theory of quantales in studying M. We shall identify Q, M, and $\mathfrak{Q}(M)$ with subsets of $\Lambda(M)$ in the obvious way:

$$Q \cong \begin{bmatrix} Q^{\smile} & 0 \\ 0 & 0 \end{bmatrix}, \qquad M \cong \begin{bmatrix} 0 & 0 \\ M^{\sim} & 0 \end{bmatrix}, \qquad \mathfrak{Q}(M) \cong \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{Q}(M) \end{bmatrix}$$

After this identification, the module multiplication $\diamond: M \times Q \to M$ becomes a part of the internal multiplication of $\Lambda(M)$; namely,

$$\begin{bmatrix} 0 & 0 \\ m^{\sim} & 0 \end{bmatrix} \circ \begin{bmatrix} a^{\smile} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (m \diamond a)^{\sim} & 0 \end{bmatrix}$$

By Theorem 1.6, we have a quantale representation $\pi: \Lambda(M) \to \mathfrak{Q}(S)$ such that S is a Girard bimodule over the quantale $\Lambda(M)$. Then by restriction π defines two maps $\varphi = \pi|Q$ and $\Phi = \pi|M$ which together constitute a representation of the right Q-module M. ■

Corollary 2.4. Any right (left) Q-module M can be equipped with a structure of a right (left) Q'-valued module such that Q is a subquantale of Q'.

Proof. It follows from Theorem 2.3 if we put $Q' = \Lambda(M)$.

Theorem 2.5. Let Q be a unital quantale, M a right Q-valued module. Then we have an inner product-preserving representation (Φ, φ) such that S is a Girard bimodule over Q.

Proof. As before, we shall show that any right Q-valued module M always can be embedded into a the quantale $\Lambda(M)$ from Theorem 2.3. Then $N = Q \times M$ is equipped with the inner product

$$\langle (a, m), (b, n) \rangle = a \cdot b \vee \langle m, n \rangle$$

We can identify each element $m \in M$ with the operator $m^*: M \to Q$ defined by $n \mapsto \langle m, n \rangle$. We may identify M with a subset of $\Lambda(M)$ in the obvious way:

$$M \cong \begin{bmatrix} 0 & M^* \\ 0 & 0 \end{bmatrix}$$

This subset is called the *conjugate module* of M and is denoted by M^*. We then put $\Phi^* = \pi|M^*$. The Q-valued inner product $\langle m, n \rangle$ of M becomes simply the product $m^* \circ n^{\sim}$. Namely, we have

$$\begin{bmatrix} 0 & m^* \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 \\ n^{\sim} & 0 \end{bmatrix} = \begin{bmatrix} \langle m, n \rangle^{\smile} & 0 \\ 0 & 0 \end{bmatrix}$$

The rest follows from Theorem 2.3. ■

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